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Longitudinal correlation function inequalities for the Ising-like n-vector model

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Abstract. For a completely anisotropic n-vector model with Ising-like (pure single component) coupling and non-negative single-spin and pair-spin interactions, we establish the longitudinal correlation function inequalities

$$\langle S_1^\delta \rangle \geqslant 0; \qquad \langle S_1^\delta S_1^\nu \rangle - \langle S_1^\delta \rangle \langle S_1^\nu \rangle \geqslant 0;$$

where the spins $\boldsymbol{S}_i = (S_{i1}, S_{i2}, \ldots, S_{in})$ and $S_1^\delta = \Pi_{i=1}^N (S_{i1})^{\delta_i}$ with δ a multiplicity function assigning a non-negative integer to each site. In particular for $n = 1,2$, and 3 respectively, inequalities are recovered for a spin-$\frac{1}{2}$ Ising model, an anisotropic planar classical Heisenberg model, and a spin-∞ Ising model.

The success of the GKS (Griffiths 1967, Kelly and Sherman 1968) inequalities for Ising models has evoked a concerted effort to extend these correlation inequalities to other 'ferromagnetic' systems (Griffiths 1974). Recent advances have been made in this direction for continuous spin systems. In particular, Monroe (1975) has established GKS type inequalities for the anisotropic planar classical Heisenberg model. Unfortunately his methods appear to fail for the three-dimensional Heisenberg model, so the ultimate problem of GKS inequalities for the n-vector model remains outstanding.

In this paper we will be concerned with the Ising-like n-vector model. The Hamiltonian is

$$\mathscr{H} = -\sum_{1 \leqslant i < j \leqslant N} J_{ij} S_{i1} S_{j1} - \sum_{i=1}^{N} H_i S_{i1} \tag{1}$$

where the spins $\boldsymbol{S}_i = (S_{i1}, S_{i2}, \ldots, S_{in})$, $i = 1, 2, \ldots, N$ have norm $\|\boldsymbol{S}_i\| = n^{1/2}$ and $n = 1$, 2 and 3 corresponds respectively to a spin-$\frac{1}{2}$ Ising model, an extremely anisotropic planar classical Heisenberg model and, as pointed out by Moore *et al* (1974), the spin-∞ Ising model. Several authors (Moore *et al* 1974, Aharony 1974, Hikami 1974) have recently investigated this model in connection with the crossover from Ising to mean-field behaviour in the spherical ($n \to \infty$) limit.

Since no correlation inequalities have yet been proved for the n-vector model it is of interest, and our intention here, to establish GKS inequalities for the Ising-like n-vector model. More specifically for products of longitudinal components

$$S_1^\delta = \prod_{i=1}^{N} (S_{i1})^{\delta_i},$$

where δ is a multiplicity function assigning a non-negative integer to each site, we assert that

$$\langle S_1^\delta \rangle \geqslant 0 \tag{2}$$

$$\langle S_1^\delta S_1^\nu \rangle - \langle S_1^\delta \rangle \langle S_1^\nu \rangle \geqslant 0 \tag{3}$$

under the conditions that

$$J_{ij} \geqslant 0, \qquad 1 \leqslant i < j \leqslant N \qquad H_i \geqslant 0, \qquad 1 \leqslant i \leqslant N. \tag{4}$$

In particular the second inequality implies the monotonicity results:

$$\frac{\partial \langle S_1^\delta \rangle}{\partial J_{ij}} \geqslant 0, \qquad \frac{\partial \langle S_1^\delta \rangle}{\partial H_i} \geqslant 0.$$

We remark further that for $n = 1, 2$ and 3 these inequalities are not new (Griffiths 1967, 1969, Monroe 1975) but emphasize that the $n = 3$ inequalities, previously proved in the guise of the spin-∞ Ising model, become in this context the first correlation inequalities for a true, though extremely anisotropic classical Heisenberg model.

The proof of the first inequality (2) is straightforward (following Monroe 1975) and will be omitted. To prove the second inequality (3) we follow the approach of Monroe and Siegert (1974) based on the identity

$$\exp\left(\tfrac{1}{2} \sum_{i,j=1}^{N} \alpha_{ij} \xi_i \xi_j\right) = (2\pi)^{-N/2} (\det \alpha)^{-1/2} \int_{-\infty}^{\infty} \dots \int \prod_{i=1}^{N} \mathrm{d}x_i \exp\left(-\tfrac{1}{2} \sum_{i,j=1}^{N} (\alpha^{-1})_{ij} x_i x_j + \sum_{i=1}^{N} x_i \xi_i\right), \tag{5}$$

valid for any positive definite real symmetric matrix α and for any N variables ξ_i, and the observation that if the elements of α are non-negative then

$$(2\pi)^{-N/2} (\det \alpha)^{-1/2} \int_{-\infty}^{\infty} \dots \int \prod_{i=1}^{N} \mathrm{d}x_i \left(\prod_{k=1}^{N} x_k^{n_k}\right) \exp\left(-\tfrac{1}{2} \sum_{i,j=1}^{N} (\alpha^{-1})_{ij} x_i x_j\right) \geqslant 0 \tag{6}$$

for any choice of the non-negative integers n_k.

By duplicating the spin variables and then using the identity (5) we obtain

$$\begin{aligned}
2Z^2(\langle S_1^\delta S_1^\nu \rangle - \langle S_1^\delta \rangle \langle S_1^\nu \rangle) &= \int_{\|S_i^1\| = \|S_i^2\| = n^{1/2}} \dots \int \mathrm{d}^N \boldsymbol{S}^1 \, \mathrm{d}\boldsymbol{S}^2 ((S_1^1)^\delta - (S_1^2)^\delta)((S_1^1)^\nu - (S_1^2)^\nu) \\
&\quad \times \exp\left(\tfrac{1}{2} \sum_{i \neq j} J_{ij} (S_{i1}^1 S_{j1}^1 + S_{i1}^2 S_{j1}^2) + \sum_{i=1}^{N} H_i (S_{i1}^1 + S_{i1}^2)\right) \\
&= (2\pi)^{-N} (\det J)^{-1} \int_{-\infty}^{\infty} \dots \int \mathrm{d}^N x \, \mathrm{d}^N y \exp\left(-\tfrac{1}{2} \sum_{i,j=1}^{N} (J^{-1})_{ij} (x_i x_j + y_i y_j)\right) \\
&\quad \times \left(\prod_{i=1}^{N} (\partial/\partial x_i)^{\delta_i} - \prod_{i=1}^{N} (\partial/\partial y_i)^{\delta_i}\right)\left(\prod_{i=1}^{N} (\partial/\partial x_i)^{\nu_i} - \prod_{i=1}^{N} (\partial/\partial y_i)^{\nu_i}\right) \\
&\quad \times \int_{\|S_i^1\| = \|S_i^2\| = n^{1/2}} \dots \int \mathrm{d}^N \boldsymbol{S}^1 \, \mathrm{d}^N \boldsymbol{S}^2 \prod_{i=1}^{N} \exp[-J_0((S_{i1}^1)^2 + (S_{i1}^2)^2) \\
&\quad + (x_i + H_i) S_{i1}^1 + (y_i + H_i) S_{i1}^2]
\end{aligned}$$

where we choose the diagonal elements $J_{ii}=J_0$ large enough to guarantee that the symmetric matrix J is positive definite. Performing the orthogonal transformations to new variables

$$\xi_i = 1/\sqrt{2}(x_i+y_i), \qquad \eta_i = 1/\sqrt{2}(x_i-y_i)$$

gives

$$\begin{aligned}
&2Z^2(\langle S_1^\delta S_1^\nu\rangle - \langle S_1^\delta\rangle\langle S_1^\nu\rangle) \\
&\quad = (2\pi)^{-N}(\det J)^{-1}\int_{-\infty}^{\infty}\dots\int \mathrm{d}^N\xi\,\mathrm{d}^N\eta\Big(\exp -\tfrac{1}{2}\sum_{i,j}(J^{-1})_{ij}(\xi_i\xi_j+\eta_i\eta_j)\Big) \\
&\quad \times\Big(\prod_{i=1}^N (1/\sqrt{2})^{\delta_i}(\partial/\partial\xi_i+\partial/\partial\eta_i)^{\delta_i} - \prod_{i=1}^N (1/\sqrt{2})^{\delta_i}(\partial/\partial\xi_i-\partial/\partial\eta_i)\Big) \\
&\quad \times\Big(\prod_{i=1}^N (1/\sqrt{2})^{\nu_i}(\partial/\partial\xi_i+\partial/\partial\eta_i)^{\nu_i} - \prod_{i=1}^N (1/\sqrt{2})^{\nu_i}(\partial/\partial\xi_i-\partial/\partial\eta_i)^{\nu_i}\Big) \\
&\quad \times \int_{\|S_i^1\|=\|S_i^2\|=n^{1/2}}\dots\int \mathrm{d}^N\boldsymbol{S}^1\,\mathrm{d}^N\boldsymbol{S}^2\prod_{i=1}^N \exp[-J_0((S_{i1}^1)^2+(S_{i1}^2)^2)] \\
&\quad \times\prod_{i=1}^N \exp\{[(1/\sqrt{2})\xi_i+H_i](S_{i1}^1+S_{i1}^2)+(1/\sqrt{2})\eta_i(S_{i1}^1-S_{i1}^2)\}. \qquad (7)
\end{aligned}$$

The remainder of the proof is devoted to showing that the right-hand side of (7) reduces to a sum of non-negative terms. Firstly the negative terms of the derivatives cancel and we can write

$$\begin{aligned}
&\prod_{i=1}^N (\partial/\partial\xi_i+\partial/\partial\eta_i)^{\delta_i} - \prod_{i=1}^N (\partial/\partial\xi_i-\partial/\partial\eta_i)^{\delta_i} \\
&\quad = \sum_{\substack{p_1,q_1,\dots,p_N,q_N\\ \sum_i q_i \text{ odd}}} C(p_1,q_1,\dots,p_N,q_N)\prod_{i=1}^N (\partial/\partial\xi_i)^{p_i}(\partial/\partial\eta_i)^{q_i} \qquad (8)
\end{aligned}$$

where the expansion coefficients $C(p_1, q_1, \dots, p_N, q_N)$ are non-negative. Hence by the observation (6) and the conditions (4) all that remains to be shown is that the configurational integrals in (7) have a series in the ξ, η and H variables with non-negative coefficients, since then the result is established by performing the differentiations (8) followed by the Gaussian integrals (6) term by term in (7).

By expanding the second product of exponentials in (7) the coefficients of interest are seen to be products of integrals of the form

$$I_{pq} = \int_{\|S^1\|=\|S^2\|=n^{1/2}}\dots\int \mathrm{d}\boldsymbol{S}^1\,\mathrm{d}\boldsymbol{S}^2\exp[-J_0((S_1^1)^2+(S_1^2)^2](S_1^1+S_1^2)^p(S_1^1-S_1^2)^q.$$

By the symmetries of the integral I_{pq} we have $I_{pq}=0$ unless both the integers p and q are even. We therefore conclude that $I_{pq}\geqslant 0$ and that all the coefficients in the expansion of (7) are non-negative as required for the completion of the proof.

In conclusion we comment that it has been shown (Pearce and Thompson 1976) that the free energy of the Ising-like n-vector model is identical to that of the mean-field spherical model in the spherical ($n\to\infty$) limit. If the correlations also carry over in this limit our results would then imply GKS inequalities for the mean-field spherical model, which would be an exceptional case in view of the recent observation (Pearce 1976) that GKS inequalities are not in general valid for the spherical model.

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